Published by Institute of Physics Publishing for SISSA

Received: *January 17, 2008*
Accepted: *January 22, 2008*
Published: *January 28, 2008*

Explaining the pure spinor formalism for the superstring

Nathan Berkovits

Instituto de Física Teórica, State University of São Paulo,
Rua Pamplona 145, 01405-900, São Paulo, SP, Brasil
E-mail: nberkovi@ift.unesp.br

Abstract: After adding a pair of non-minimal fields and performing a similarity transformation, the BRST operator in the pure spinor formalism is expressed as a conventional-looking BRST operator involving the Virasoro constraint and (b, c) ghosts, together with 12 fermionic constraints. This BRST operator can be obtained by gauge-fixing the Green-Schwarz superstring where the 8 first-class and 8 second-class Green-Schwarz constraints are combined into 12 first-class constraints. Alternatively, the pure spinor BRST operator can be obtained from the RNS formalism by twisting the ten spin-half RNS fermions into five spin-one and five spin-zero fermions, and using the SO(10)/ U(5) pure spinor variables to parameterize the different ways of twisting. $GSO(-)$ vertex operators in the pure spinor formalism are constructed using spin fields and picture-changing operators in a manner analogous to Ramond vertex operators in the RNS formalism.

Keywords: Superstrings and Heterotic Strings, Topological Strings.

JHEP01(2008)065

Contents

1. Introduction

The pure spinor formalism [1] is a super-Poincaré covariant description of the superstring which significantly simplifies multiloop amplitude computations and which allows quantization in Ramond-Ramond backgrounds. However, because of the non-conventional form of the BRST operator in the pure spinor formalism, the relation of this formalism to the Green-Schwarz (GS) and Ramond-Neveu-Schwarz (RNS) formalisms for the superstring was mysterious. Furthermore, it was not known how to describe the $GSO(-)$ sector of the superstring using the pure spinor formalism.

In this paper, these mysterious features of the formalism will be explained by adding a pair of non-minimal fields and performing a similarity transformation such that the

JHEP01(2008)065

pure spinor BRST operator is expressed as a conventional-looking BRST operator. This conventional-looking BRST operator involves the Virasoro constraint and twelve fermionic constraints, where eleven of these fermionic constraints are associated to the eleven independent components of the original bosonic pure spinor ghost. The twelfth fermionic constraint and the Virasoro constraint are associated to the new pair of non-minimal fields, $(\widetilde{\beta}, \widetilde{\gamma})$ and (b, c), which have opposite statistics and carry conformal weight $(2, -1)$. Although this conventional form of the BRST operator is not manifestly Lorentz invariant, it will be useful for constructing $GSO(-)$ vertex operators and for relating the pure spinor formalism to the GS and RNS formalisms.

The new non-minimal fields, $(\widetilde{\beta}, \widetilde{\gamma})$ and (b, c), decouple from vertex operators and scattering amplitudes involving $GSO(+)$ states, however, they play a crucial role in defining vertex operators and scattering amplitudes involving $GSO(-)$ states. Just as Ramond vertex operators in the RNS formalism [2] depend non-trivially on the (β, γ) ghosts, $GSO(-)$ vertex operators in the pure spinor formalism will depend non-trivially on the $(\widetilde{\beta}, \widetilde{\gamma})$ ghosts. And just as scattering amplitudes involving Ramond states in the RNS formalism require picture-changing operators to cancel the picture of the Ramond vertex operators, scattering amplitudes involving $GSO(-)$ states in the pure spinor formalism will require picture-changing operators to cancel the picture of the $GSO(-)$ vertex operators.

Note that the new non-minimal fields $(\widetilde{\beta}, \widetilde{\gamma}, b, c)$ are unrelated to the non-minimal fields $(\overline{\lambda}_\alpha, \overline{w}^\alpha, r_\alpha, s^\alpha)$ which were introduced in the "Dolbeault" description of the pure spinor formalism [3, 4] [5]. In this paper, the Dolbeault description will not be discussed although it would be interesting to consider including both $(\widetilde{\beta}, \widetilde{\gamma}, b, c)$ and $(\overline{\lambda}_\alpha, \overline{w}^\alpha, r_\alpha, s^\alpha)$ non-minimal fields in the pure spinor formalism. Such a Dolbeault description might be useful for writing the conventional-looking BRST operator in a manifestly Lorentz-invariant form.

After expressing the pure spinor BRST operator as a conventional-looking BRST operator with a Virasoro constraint and twelve fermionic constraints, it is relatively straightforward to relate the pure spinor formalism with the GS and RNS formalisms for the superstring. In the GS formalism, the fermionic constraint $d_\alpha = 0$ contains 8 first-class components and 8 second-class components. After breaking manifest Lorentz invariance down to SO(8) and then to U(4), the 8 second-class constraints can be converted into 4 first-class constraints. The resulting BRST operator has 12 fermionic constraints and is related by a field redefinition to the pure spinor BRST operator. Interestingly, this field redefinition allows the manifest U(4) invariance to be enlarged to U(5).[1]

To relate the RNS formalism with the pure spinor formalism, one first twists the ten spin-half RNS fermions ψ^m into five spin-zero fermions θ^a and five spin-one fermions p_a for $a = 1$ to 5.[2] This twisting breaks SO(10) Lorentz invariance to U(5), and one can parameterize the different choices of twisting by introducing SO(10)/ U(5) bosonic pure

[1]Throughout this paper, we shall Wick-rotate the spacetime metric from SO(9, 1) to SO(10). All results can be Wick-rotated back to Minkowski space, however, the group structure of the 25 U(5) generators is more complicated in Minkowski space where it splits into 16 U(4) generators and 9 light-like boosts.

[2]A similar twisting procedure was used in several earlier papers by Baulieu and collaborators to relate the RNS string to a topological string [6]. I thank Nikita Nekrasov for informing me of these papers.

spinor variables. One then imposes the constraints that physical states are independent of the 11 pure spinor variables, and the fermionic ghosts for these constraints are the remaining 11 components of θ^α and p_α. After adding these 11 constraints to the N=1 super-Virasoro constraints, the RNS BRST operator is mapped into the conventional form of the pure spinor BRST operator where the spin -1 non-minimal field $\widetilde{\gamma}$ is related to the spin $-\frac{1}{2}$ RNS ghost γ as $\widetilde{\gamma} = (\gamma)^2$.

It is interesting to note that a similar procedure of twisting fermions has been used to embed the $N = 0$ bosonic string into an $N = 1$ string [7]. In the $N = 0 \to N = 1$ embedding, the (b, c) ghosts are twisted from $(2, -1)$ conformal weight to $(\frac{3}{2}, -\frac{1}{2})$ conformal weight, and the $N = 1$ stress tensor is defined as $G = b + j_{\mathrm{BRST}}$ where $\int dz\ j_{\mathrm{BRST}}$ is the BRST charge of the bosonic string. In fact, the inverse map of this embedding which takes an $N = 1$ string into an $N = 0$ string is closely related to the map from the RNS formalism to the pure spinor formalism. This is not surprising since the pure spinor formalism can be interpreted as an $N = 2$ topological string [3, 8], which is a natural generalization of $N = 0$ bosonic strings.

Note that this $N = 1 \to N = 0$ inverse map from the RNS to the pure spinor formalism is different from the $N = 1 \to N = 2$ embedding that has been used to map the RNS formalism into the hybrid formalism for the superstring [9]. For example, the $N = 1 \to N = 2$ embedding maps the RNS string into a critical $\hat{c} = 2$ $N = 2$ string as opposed to a $\hat{c} = 3$ $N = 2$ topological string. Nevertheless, a certain version of the $N = 1 \to N = 2$ embedding will be shown at the end of this paper to closely resemble the $N = 1 \to N = 0$ embedding. This version might eventually be useful for relating the pure spinor and hybrid formalisms for the superstring.

In section 2 of this paper, the pure spinor formalism is briefly reviewed and a pair of non-minimal fields, (b, c) and $(\widetilde{\beta}, \widetilde{\gamma})$, are introduced. After performing a similarity transformation, the pure spinor BRST operator is expressed as a conventional-looking BRST operator with 12 fermionic constraints.

In section 3, $GSO(-)$ vertex operators are constructed with the help of the non-minimal fields. These $GSO(-)$ vertex operators carry nonzero picture and, after defining picture-changing operators, it is shown how to compute scattering amplitudes using these vertex operators.

In section 4, the conventional-looking form of the pure spinor BRST operator is obtained from gauge-fixing the GS superstring. In performing this gauge-fixing, the 8 first-class and 8 second-class GS constraints are combined into 12 first-class constraints in a manifestly U(5)-invariant manner.

In section 5, the RNS BRST operator is mapped to the pure spinor BRST operator by twisting the ten spin-half RNS fermions using an SO(10)/ U(5) pure spinor variable to parameterize the different twistings. For states in the Neveu-Schwarz $GSO(+)$ sector, it is shown how to map the RNS and pure spinor vertex operators into each other.

In section 6, the map from the RNS formalism to the pure spinor formalism is interpreted as an inverse map of the $N = 0 \to N = 1$ embedding of the bosonic string. This inverse map may be useful for constructing generalizations of the pure spinor formalism.

And in section 7, the approach of this paper will be compared with other approaches to "explaining" the pure spinor formalism. One approach which is discussed in detail uses an $N=1 \to N=2$ embedding to map the RNS string into variables which closely resemble those of the $N=1 \to N=0$ embedding.

2. Conventional-looking pure spinor BRST operator

In this section, the pure spinor formalism will be briefly reviewed and the BRST operator $Q = \int dz \lambda^\alpha d_\alpha$ will be related to a conventional-looking BRST operator involving the usual (b,c) ghosts and Virasoro constraint, together with 12 fermionic constraints.

2.1 Brief review of pure spinor formalism

The pure spinor formalism [1] in a flat background is described by the free worldsheet action

$$S = \int d^2 z \left[\frac{1}{2} \partial x^m \overline{\partial} x_m + p_\alpha \overline{\partial} \theta^\alpha + \widehat{p}_{\widehat{\alpha}} \partial \widehat{\theta}^{\widehat{\alpha}} + w_\alpha \overline{\partial} \lambda^\alpha + \widehat{w}_{\widehat{\alpha}} \partial \widehat{\lambda}^{\widehat{\alpha}} \right] \tag{2.1}$$

where $(x^m, \theta^\alpha, \widehat{\theta}^{\widehat{\alpha}}, p_\alpha, \widehat{p}_{\widehat{\alpha}})$ are the Green-Schwarz-Siegel matter variables for $m=0$ to 9 and $(\alpha, \widehat{\alpha}) = 1$ to 16, $(\lambda^\alpha, w_\alpha)$ and $(\widehat{\lambda}^{\widehat{\alpha}}, \widehat{w}_{\widehat{\alpha}})$ are left and right-moving bosonic ghost variables satisfying the pure spinor constraint

$$\lambda \Gamma^m \lambda = \widehat{\lambda} \Gamma^m \widehat{\lambda} = 0, \tag{2.2}$$

and $\Gamma^m_{\alpha\beta}$ and $(\Gamma^m)^{\alpha\beta}$ are 16×16 symmetric matrices satisfying $\Gamma^{(m}_{\alpha\beta} (\Gamma^{n)})^{\beta\gamma} = 2 \delta^\gamma_\alpha \eta^{mn}$. The hatted spinor variables have the opposite/same chirality as the unhatted variables for the Type IIA/IIB superstring, and throughout this paper, the hatted variables will be ignored.

Physical states are defined as states in the cohomology of the BRST operator

$$Q = \int dz\ \lambda^\alpha d_\alpha \tag{2.3}$$

where

$$d_\alpha = p_\alpha - \frac{1}{2} (\Gamma^m \theta)_\alpha \partial x_m - \frac{1}{8} (\theta \Gamma^m \partial \theta)(\Gamma_m \theta)_\alpha \tag{2.4}$$

is the Green-Schwarz constraint. Since d_α satisfies the OPE [10]

$$d_\alpha(y) d_\beta(z) \to -(y-z)^{-1} \Gamma^m_{\alpha\beta} \Pi_m \tag{2.5}$$

where $\Pi_m = \partial x_m + \frac{1}{2} \theta \Gamma_m \partial \theta$ is the supersymmetric momentum, Q is nilpotent using the constraint of (2.2).

For massless super-Yang-Mills states, the unintegrated and integrated vertex operators are

$$V = \lambda^\alpha A_\alpha(x,\theta), \tag{2.6}$$

$$\int dz\ U = \int dz [\partial \theta^\alpha A_\alpha(x,\theta) + \Pi^m A_m(x,\theta) + d_\alpha W^\alpha(x,\theta) + N_{mn} F^{mn}(x,\theta)] \tag{2.7}$$

where $N^{mn} = \frac{1}{2}(w\Gamma^{mn}\lambda)$ is the Lorentz current for the pure spinor variables, (A_α, A_m) are gauge superfields and (W^α, F^{mn}) are superfield-strengths for super-Yang-Mills. When the super-Yang-Mills superfields are onshell, $QV = 0$ and $QU = \partial V$.

Tree-level N-point scattering amplitudes are computed by the correlation function

$$\mathcal{A} = \langle\langle V_1 V_2 V_3 \int dz_4 U_4 \ldots \int dz_N U_N \rangle\rangle \tag{2.8}$$

using the measure factor

$$\langle\langle (\lambda\Gamma^m\theta)(\lambda\Gamma^n\theta)(\lambda\Gamma^p\theta)(\theta\Gamma_{mnp}\theta) \rangle\rangle = 1. \tag{2.9}$$

Although this measure factor looks unusual, it can be derived from functional integration over the worldsheet fields after performing a BRST-invariant regularization [3].

The correlation function of (2.8) is easily computed using the free-field OPE's coming from the worldsheet action of (2.1) together with the OPE's

$$N^{mn}(y)\lambda^\alpha(z) \to \frac{1}{2}(y-z)^{-1}(\Gamma^{mn}\lambda)^\alpha, \tag{2.10}$$

$$N^{mn}(y)N^{pq}(z) \to (y-z)^{-1}(\eta^{p[n}N^{m]q} - \eta^{q[n}N^{m]p}) - 3(y-z)^{-2}\eta^{m[q}\eta^{p]n}. \tag{2.11}$$

The manifestly covariant OPE's of (2.10) can be derived by solving the pure spinor constraint $\lambda\Gamma^m\lambda = 0$ in a U(5)-invariant manner. Under SU(5) $\times$ U(1), an SO(10) spinor decomposes as $\lambda^\alpha \to (\lambda^+, \lambda_{ab}, \lambda^a)$ where $a = 1$ to 5, $\lambda_{ab} = -\lambda_{ba}$, and $(\lambda^+, \lambda_{ab}, \lambda^a)$ carries U(1) charge $(\frac{5}{2}, \frac{1}{2}, -\frac{3}{2})$. If λ^+ is assumed to be nonzero, $\lambda\Gamma^m\lambda = 0$ implies that

$$\lambda^a = -\frac{1}{8}(\lambda^+)^{-1}\epsilon^{abcde}\lambda_{bc}\lambda_{de} \tag{2.12}$$

so that λ^α has eleven independent components parameterized by λ^+ and λ_{ab}.

In terms of $(\lambda^+, \lambda_{ab})$ and their conjugate momenta (w_+, w^{ab}), the pure spinor contribution to the stress tensor and Lorentz currents is [11]

$$T_{\rm pure} = \frac{1}{2}w^{ab}\partial\lambda_{ab} + w_+\partial\lambda^+ + \frac{3}{2}\partial^2(\log\ \lambda^+), \tag{2.13}$$

$$N_{\rm U(1)} = \frac{1}{\sqrt{5}}\left(\frac{1}{4}\lambda_{ab}w^{ab} + \frac{5}{2}\lambda^+ w_+ - \frac{5}{4}\partial(\log\ \lambda^+)\right), \tag{2.14}$$

$$N^{ab} = \lambda^+ w^{ab}, \tag{2.15}$$

$$N_a^b = \lambda_{ac}w^{bc} - \frac{1}{5}\delta_a^b\lambda_{cd}w^{cd},$$

$$N_{ab} = (\lambda^+)^{-1}\left(2\partial\lambda_{ab} - \frac{5}{2}\lambda_{ab}\partial(\log\ \lambda^+) + \lambda_{ac}\lambda_{bd}w^{cd} - \frac{1}{2}\lambda_{ab}\lambda_{cd}w^{cd}\right) - w_+\lambda_{ab}, \tag{2.16}$$

where the SO(10) Lorentz currents N^{mn} have been decomposed into $(N_{\rm U(1)}, N_a^b, N^{ab}, N_{ab})$ which transform as $(1, 24, 10, \overline{10})$ representations of SU(5). Note that the "improvement" term $\frac{3}{2}\partial^2(\log\lambda^+)$ is necessary in $T_{\rm pure}$ so that N_{mn} are primary fields with respect to $T_{\rm pure}$. It is also convenient to define the ghost-current

$$J = w_+\lambda^+ + \frac{1}{2}w^{ab}\lambda_{ab} + \frac{7}{2}\partial(\log\ \lambda^+) \tag{2.17}$$

which has no poles with N^{mn} and which satisfies $J(y)\lambda^\alpha(z) \to (y-z)^{-1}\lambda^\alpha$.

Although there is no fundamental b ghost in the pure spinor formalism, one can construct a composite operator G^α satisfying $\{Q, G^\alpha\} = \lambda^\alpha T$ where

$$T = -\frac{1}{2}\partial x^m \partial x_m - p_\alpha \partial\theta^\alpha + T_{\text{pure}} \tag{2.18}$$

is the stress tensor with zero central charge.[3] This composite operator will play an important role in this paper and is defined as [11]

$$G^\alpha = \frac{1}{2}\Pi^m(\Gamma_m d)^\alpha - \frac{1}{4}N^{mn}(\Gamma_{mn}\partial\theta)^\alpha - \frac{1}{4}J\partial\theta^\alpha - \frac{1}{4}\partial^2\theta^\alpha \tag{2.19}$$

where N_{mn} and J are defined in (2.14) and (2.17).

2.2 Non-minimal fields and similarity transformation

The first step to constructing a conventional-looking BRST operator from $Q = \int dz \lambda^\alpha d_\alpha$ is to add the term $\int dz \widetilde{\gamma} b$ to the pure spinor BRST operator so that

$$Q = \int dz(\lambda^\alpha d_\alpha + \widetilde{\gamma} b) \tag{2.20}$$

where $(\widetilde{\beta}, \widetilde{\gamma})$ are bosonic and (b, c) are fermionic non-minimal fields with the worldsheet action $\int d^2z(\widetilde{\beta}\overline{\partial}\widetilde{\gamma} + b\overline{\partial}c)$. These non-minimal fields do not contribute to the cohomology because of the topological term $\int dz\widetilde{\gamma}b$ in Q.

The second step is to perform the similarity transformation $Q' = e^R Q e^{-R}$ where

$$R = \oint dz\left[-\frac{c}{\lambda^+}G^+ + c\partial c\widetilde{\beta}\right] \tag{2.21}$$

and G^+ is the component of G^α in (2.19) with $\frac{5}{2}$ U(1) charge. Using $\{Q, G^+\} = \lambda^+ T$, it is easy to verify that after performing the similarity transformation,

$$Q' = e^R Q e^{-R} = \int dz\left[c\widetilde{T} - \frac{\widetilde{\gamma}}{\lambda^+}G^+ + \lambda^\alpha d_\alpha + \widetilde{\gamma}b + bc\partial c\right] \tag{2.22}$$

where

$$\widetilde{T} = -\frac{1}{2}\partial x^m \partial x_m - p_\alpha\partial\theta^\alpha + T_{\text{pure}} + \widetilde{\beta}\partial\widetilde{\gamma} + \partial(\widetilde{\beta}\widetilde{\gamma}) \tag{2.23}$$

is a stress tensor with central charge $c = 26$.

Although Q' is not invariant under Lorentz transformations generated by M_{ab} which transform λ^+ and G^+ into λ_{ab} and G_{ab}, one can use the relation [3]

$$\lambda^{[\alpha}G^{\beta]} = \left[\int dz\ \lambda^\gamma d_\gamma,\ H^{\alpha\beta}\right] \tag{2.24}$$

[3] It is interesting to point out that in a curved target-space background, G^α will in general not be holomorphic. Nevertheless, one can argue that $\overline{\partial}G^\alpha$ is BRST-trivial, which appears to be sufficient for computing scattering amplitudes where G^α plays the role of the b ghost.

where

$$H^{\alpha\beta} = \frac{1}{192}\Gamma^{\alpha\beta}_{mnp}(d\Gamma^{mnp}d + 24N^{mn}\Pi^p) \tag{2.25}$$

to show that

$$\left[\, Q',\ M_{ab} + \int dz \frac{\widetilde{\gamma}}{(\lambda^+)^2} H^+_{ab} \,\right] = 0 \tag{2.26}$$

where H^+_{ab} is the component of $H^{\alpha\beta}$ with $\alpha = +$ and $\beta = ab$. Furthermore, one can verify that the Lorentz algebra generated by $M'_{ab} \equiv M_{ab} + \int dz \frac{\widetilde{\gamma}}{(\lambda^+)^2} H^+_{ab}$ with the other Lorentz generators closes up to a BRST-trivial operator. So under Lorentz transformations generated by $M'_{mn} = [M_{\mathrm{U}(1)}, M^{ab}, M^a_b, M_{ab} + \int dz \frac{\widetilde{\gamma}}{(\lambda^+)^2} H_{ab}]$, Q'-closed states transform covariantly up to a BRST-trivial transformation. Note that one could have defined the Lorentz generators as $M'_{mn} = e^R M_{mn} e^{-R}$ where R is defined in (2.21), but such a definition would not preserve the property that all poles when $\lambda^+ \to 0$ have residues which are proportional to $\widetilde{\gamma}$. As will be discussed later, this property is useful since terms proportional to $\widetilde{\gamma}$ will decouple from scattering amplitudes.

Finally, it will be convenient to define

$$\widetilde{\gamma}_+ = -\frac{\widetilde{\gamma}}{\lambda^+}, \tag{2.27}$$

so that

$$Q' = \int dz[c\widetilde{T} + \widetilde{\gamma}_+ G^+ + \lambda^\alpha d_\alpha - \lambda^+\widetilde{\gamma}_+ b + bc\partial c]. \tag{2.28}$$

If $(\widetilde{\gamma}_+, \lambda^+, \lambda_{ab})$ are interpreted as 12 independent bosonic ghosts, Q' resembles a standard BRST operator constructed from 12 fermionic constraints and the Virasoro constraint.

Since $(\widetilde{\gamma}_+, \widetilde{\beta}^+)$ are not Lorentz scalars, they will appear in the Lorentz generators. In terms of $(\widetilde{\gamma}_+, \widetilde{\beta}^+)$, (w_+, λ^+) and (w^{ab}, λ_{ab}), the SO(10) Lorentz currents of (2.14) are

$$N_{\mathrm{U}(1)} = \frac{1}{\sqrt{5}}\left(\frac{1}{4}\lambda_{ab}w^{ab} + \frac{5}{2}\lambda^+ w_+ - \frac{5}{2}\widetilde{\gamma}_+\widetilde{\beta}^+\right), \tag{2.29}$$

$$N^{ab} = \lambda^+ w^{ab}, \quad N^b_a = \lambda_{ac}w^{bc} - \frac{1}{5}\delta^b_a \lambda_{cd}w^{cd}, \tag{2.30}$$

$$N_{ab} = (\lambda^+)^{-1}\left(2\partial\lambda_{ab} + \lambda_{ab}\widetilde{\gamma}_+\widetilde{\beta}^+ - 4\lambda_{ab}\partial(\log\ \lambda^+) + \lambda_{ac}\lambda_{bd}w^{cd} - \frac{1}{2}\lambda_{ab}\lambda_{cd}w^{cd}\right) - w_+\lambda_{ab}. \tag{2.31}$$

The contribution of these bosonic ghosts to the stress tensor is

$$\widetilde{T}_{\mathrm{pure}} = w_+\partial\lambda^+ + \frac{1}{2}w^{ab}\partial\lambda_{ab} + \widetilde{\beta}^+\partial\widetilde{\gamma}_+ + \partial(\widetilde{\beta}^+\widetilde{\gamma}_+), \tag{2.32}$$

which can be verified to have no triple poles with N^{mn}. And the ghost current of (2.17) is

$$J = w_+\lambda^+ + \frac{1}{2}w^{ab}\partial\lambda_{ab} - \widetilde{\beta}^+\widetilde{\gamma}_+ + 4\partial(\log\ \lambda^+). \tag{2.33}$$

Remarkably, after including the (b, c) and $(\widetilde{\beta}^+, \widetilde{\gamma}_+)$ non-minimal fields, $\widetilde{T}_{\mathrm{pure}}$ no longer requires improvement terms involving $\partial^2(\log\ \lambda^+)$. This may resolve some of the puzzles discussed in [12] which are related to possible anomalies in the formalism. Furthermore, as will be shown in the following section, the introduction of these non-minimal fields appears to be necessary for the construction of $GSO(-)$ vertex operators in the pure spinor formalism.

JHEP01(2008)065

3. $GSO(-)$ states in the pure spinor formalism

In this section, it will be shown how to construct vertex operators for $GSO(-)$ states and, after defining picture-changing operators, it will be shown how to compute scattering amplitudes involving these states.

3.1 $GSO(+)$ vertex operators

Before constructing $GSO(-)$ vertex operators, it will be useful to explain how to construct $GSO(+)$ vertex operators using the new BRST operator Q' of (2.28). For $GSO(+)$ states, one method to construct vertex operators V' which are BRST-invariant with respect to Q' is to simply define $V' = e^R V e^{-R}$ where R is defined in (2.21) and V is the original pure spinor vertex operator which is BRST-invariant with respect to $Q = \int dz \lambda^\alpha d_\alpha$.

However, a more useful definition is

$$V' = cU + V + \widetilde{\gamma}_+(G_0^+ U) + \frac{c\widetilde{\gamma}_+}{\lambda^+}(G_{-1}^+ G_0^+ U) \tag{3.1}$$

where V and $\int dzU$ are the original pure spinor unintegrated and integrated vertex operators satisfying $QV = 0$ and $QU = \partial V$, G_n^+ signifes the pole of order $(n+2)$ with G^+, and V has been gauge-fixed to satisfy $G_n^+ V = 0$ for $n \geq 0$. For example, for the massless super-Yang-Mills vertex operator V and U of (2.6), the gauge-fixing condition $G_0^+ V = 0$ implies that $\partial_m(\gamma^m D)^+ A_\alpha = 0$, which implies that $\partial^m \partial_m A_\alpha = \partial^m A_m = 0$.

Note that V' of (3.1) is related to $e^R V e^{-R}$ by the BRST-trivial transformation

$$V' = e^R V e^{-R} - Q'(\frac{c}{\lambda^+} G_0^+ U) \tag{3.2}$$

where the relation

$$QG_0^+ U = -G_0^+ QU + \lambda^+ T_0 U = -G_0^+ \partial V + \lambda^+ U = -\partial(G_0^+ V) + G_{-1}^+ V + \lambda^+ U \tag{3.3}$$

has been used. Although both (3.1) and $e^R V e^{-R}$ have poles when $\lambda^+ \to 0$, the vertex operator of (3.1) has the advantage that the residues of these poles are proportional to $\widetilde{\gamma}_+$. Since the vertex operators are independent of $\widetilde{\beta}^+$, any term proportional to $\widetilde{\gamma}_+$ will generically decouple from scattering amplitudes.

3.2 $GSO(-)$ vertex operators

For $GSO(-)$ states, it does not appear to be possible to construct vertex operators in the original pure spinor formalism without the non-minimal $(\widetilde{\beta}^+, \widetilde{\gamma}_+)$ fields [13] [14]. The reason is that, just as Ramond vertex operators in the RNS formalism depend non-trivially on the (β, γ) ghosts, the $GSO(-)$ vertex operators in the pure spinor formalism will depend non-trivially on the $(\widetilde{\beta}^+, \widetilde{\gamma}_+)$ ghosts.

For example, the tachyon vertex operator in the pure spinor formalism will be

$$V' = c \exp\left[-\frac{1}{2}\left(3\widetilde{\phi} + \phi^+ + \sum_{[ab]=1}^{10} \phi_{ab} - i\sum_{\alpha=1}^{16} \sigma_\alpha\right)\right] e^{ik_m x^m} \tag{3.4}$$

where the $(\widetilde{\beta}^+, \widetilde{\gamma}_+)$, (w_+, λ^+) and (w^{ab}, λ_{ab}) bosonic ghosts have been fermionized as

$$\widetilde{\beta}^+ = e^{-\widetilde{\phi}}\partial\widetilde{\xi}^+, \qquad \widetilde{\gamma}_+ = \widetilde{\eta}_+ e^{\widetilde{\phi}}, \tag{3.5}$$
$$w_+ = e^{-\phi^+}\partial\xi_+, \qquad \lambda^+ = \eta^+ e^{\phi^+}, \tag{3.6}$$
$$w^{ab} = e^{-\phi_{ab}}\partial\xi^{ab}, \qquad \lambda_{ab} = \eta_{ab} e^{\phi_{ab}}, \tag{3.7}$$

and the $(\theta^\alpha, p_\alpha)$ fields have been bosonized as

$$p_\alpha = e^{-i\sigma^\alpha}, \quad \theta^\alpha = e^{i\sigma^\alpha}.$$

Since $e^{n\widetilde{\phi}}$ carries conformal weight $\frac{1}{2}(-n^2 - 3n)$, $e^{n\phi^+}$ and $e^{n\phi^{ab}}$ carry conformal weight $\frac{1}{2}(-n^2 - n)$, and $e^{in\sigma_\alpha}$ carries conformal weight $\frac{1}{2}(n^2 - n)$, one finds that V' of (3.4) carries zero conformal weight when $e^{ik_m x^m}$ has $\frac{1}{2}$ conformal weight as expected for the tachyon. Furthermore, it is not difficult to show that $Q'V' = 0$.

Although only U(5) invariance is manifest, one can easily verify that (3.4) is a scalar under Lorentz transformations generated by (2.29). It is interesting to note that bosonized Ramond vertex operators in the RNS formalism also manifestly preserve only a U(5) subgroup of the Lorentz group.

Other $GSO(-)$ vertex operators can be constructed by taking OPE's of the tachyon vertex operator of (3.4) with the $GSO(+)$ vertex operators of (3.1). Just as (ψ^m, β, γ) have square-root cuts with Ramond vertex operators in the RNS formalism, $(\theta^\alpha, p_\alpha, \lambda^\alpha, w_\alpha, \widetilde{\beta}^+, \widetilde{\gamma}_+)$ have square-root cutes with $GSO(-)$ vertex operators in the pure spinor formalism. To be convinced that this construction of $GSO(-)$ vertex operators is correct, it will now be shown how to compute tree amplitudes using these $GSO(-)$ vertex operators.

3.3 Picture-changing operators

Because of the screening charges related to the conformal weights of the worldsheet fields, the natural measure factor for tree amplitudes is

$$\left\langle c\partial c\partial^2 c\ (\theta)^{16}\ \exp\left[-3\widetilde{\phi} - \phi^+ - \sum_{ab=1}^{10} \phi^{ab}\right] \right\rangle = 1. \tag{3.8}$$

If one defines picture such that ξ and e^ϕ carry picture $+1$ and η carries picture -1, the measure factor of (3.8) carries picture $(-3, -1, -1)$ with respect to the $(\widetilde{\gamma}_+, \lambda^+, \lambda_{ab})$ ghosts, the $GSO(+)$ vertex operators of (3.1) carry picture $(0, 0, 0)$, and the $GSO(-)$ vertex operators of (3.4) carry picture $(-\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2})$.

To relate the measure factor of (3.8) to the usual pure spinor measure factor

$$\langle\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle\rangle = 1 \tag{3.9}$$

which carries zero picture, one needs to introduce BRST-invariant picture-raising operators. As in the RNS formalism, the picture-raising operators are naturally defined by anticommuting the BRST operator with the ξ variable as

$$Z_+ = \{Q', \xi_+\} = e^{\phi^+}(d_+ - \widetilde{\gamma}_+ b) + c\partial\xi_+, \tag{3.10}$$

$$Z^{ab} = \{Q', \xi^{ab}\} = e^{\phi^{ab}}\left(d^{ab} - \frac{1}{2}(\lambda^+)^{-1}\epsilon^{abcde}\lambda_{cd}d_e\right) + c\partial\xi^{ab},$$

$$\widetilde{Z}^+ = \{Q', \widetilde{\xi}^+\} = e^{\widetilde{\phi}}(G^+ - \lambda^+ b) + c\partial\widetilde{\xi}^+. \tag{3.11}$$

By inserting products of these picture-raising operators, one finds that the measure factors of (3.8) and (3.9) can be related as

$$Z\ c\partial c\partial^2 c\ (\theta)^{16}\ e^{-3\widetilde{\phi}-\phi^+-\sum_{ab}\phi^{ab}} = (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta) + \ldots \tag{3.12}$$

where

$$\begin{aligned} Z &= (\widetilde{Z}^+)^3 Z_+ \prod_{ab} Z^{ab} \\ &= b\partial b\partial^2 b\ \epsilon^{\alpha_1\ldots\alpha_{16}}(\lambda\gamma^m)_{\alpha_1}(\lambda\gamma^n)_{\alpha_2}(\lambda\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5}\ d_{\alpha_6}\ldots d_{\alpha_{16}}\ e^{3\widetilde{\phi}+\phi^++\sum_{ab}\phi^{ab}} + \ldots, \end{aligned} \tag{3.13}$$

and $\ldots$ involves terms with fewer than three λ's (and more c's) and can be determined from the requirement of BRST invariance with respect to Q'.

3.4 Scattering amplitudes

For tree amplitudes which involve only the $GSO(+)$ vertex operators V' defined in (3.1), the N-point tree amplitude prescription is

$$A = \prod_{r=4}^{N}\int dz_r\langle\langle U_r(z_r)\prod_{s=1}^{3}V'_s(y_s)\rangle\rangle = \prod_{r=4}^{N}\int dz_r\langle\langle b(z_r)\prod_{s=1}^{N}V'_s(y_s)\rangle\rangle \tag{3.14}$$

where $\int dz_r b(z_r)$ is the usual b ghost insertion coming from the Faddeev-Popov gauge-fixing of the worldsheet action. Since there are no $\widetilde{\beta}^+$'s in this correlation function, the only terms in V' of (3.1) which contribute are $V' = cU + V$ and it is easy to verify that (3.14) reproduces the original prescription of (2.8). It is interesting that, except for the different measure factor, the prescription of (3.14) looks very similar to the Lee-Siegel prescription of [15] and it would be nice to find a proof that the two prescriptions are equivalent.

But for tree amplitudes involving $GSO(-)$ vertex operators, one needs to insert additional picture-changing operators to absorb the $(-\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2})$ picture of the $GSO(-)$ vertex operators of (3.4). This procedure is precisely analogous to RNS amplitudes involving Ramond states where the number of picture-changing operators depends on the number of Ramond vertex operators in the $-\frac{1}{2}$ picture.

For example, for tree amplitudes involving N $GSO(+)$ states V'_+ and $2M$ $GSO(-)$ states V'_-, the tree amplitude prescription is

$$\begin{aligned} A &= \prod_{r=4}^{N+2M}\int dz_r\langle\langle b(z_r)Z^M(u)\prod_{s=1}^{N}V'{}_{s+}\prod_{t=1}^{2M}V'{}_{t-}\rangle\rangle \\ &= \prod_{r=4}^{N+2M}\int dz_r\langle b(z_r)Z^{M-1}(u)\prod_{s=1}^{N}V'{}_{s+}\prod_{t=1}^{2M}V'{}_{t-}\rangle \end{aligned} \tag{3.15}$$

where the location of the picture-raising operators is arbitrary. So for tree amplitudes involving two $GSO(-)$ states and an arbitrary number of $GSO(+)$ states, one can use the natural measure factor of (3.8) without any picture-changing insertions.

For genus g amplitudes, the natural measure factor based on the screening charges is

$$\left\langle b^{3g-3}(\theta)^{16}(p)^{16g}\exp\left[(g-1)(3\widetilde{\phi}+\phi^{+}+\sum_{ab}\phi^{ab})\right]\right\rangle = 1. \tag{3.16}$$

So one expects naively that the multiloop amplitude prescription for N $GSO(+)$ states and $2M$ $GSO(-)$ states is

$$A = \prod_{j=1}^{3g-3}\int d\tau_j \prod_{r=1}^{N+2M}\int dz_r \langle\langle b(z_r)b(\mu_j)Z^{M+g}(u)\prod_{s=1}^{N}V'_{s+}\prod_{t=1}^{2M}V'_{t-}\rangle\rangle \tag{3.17}$$

where $b(\mu_j)$ is the b ghost associated with the j^{th} Teichmuller parameter τ_j. When $M=0$, this prescription appears to be closely related to the multiloop prescription given in [16] for the pure spinor formalism. However, a proof of equivalence of these multiloop prescriptions will not be attempted here.

4. Equivalence to Green-Schwarz formalism

In this section, the BRST operator Q' of (2.28) will be obtained by gauge-fixing the Green-Schwarz superstring. But before discussing the superstring, it will be useful to first discuss the Brink-Schwarz superparticle.

4.1 Brink-Schwarz superparticle

The $N=1$ $d=10$ Brink-Schwarz superparticle action, $S=\frac{1}{2}\int d\tau\ e^{-1}\Pi^m\Pi_m$, can be written in first-order form as [17, 18]

$$S = \int d\tau\left(P_m\partial_\tau x^m + p_\alpha\partial_\tau\theta^\alpha - \frac{1}{2}eP_mP^m + f^\alpha d_\alpha\right) \tag{4.1}$$

where $\Pi^m = \partial_\tau x^m + \frac{1}{2}\theta\Gamma^m\partial_\tau\theta$, $d_\alpha = p_\alpha - \frac{1}{2}P^m(\Gamma_m\theta)_\alpha$, and f^α is a fermionic Lagrange multiplier.

As is well-known, $d_\alpha = 0$ contains 8 first-class constraints and 8 second-class constraints, and the first-class constraints are generated by 8 of the 16 components of the κ-symmetry generators $P_m(\Gamma^m d)^\alpha$. One can choose $G^A = \frac{1}{2}(\Gamma^+\Gamma^m d)^A P_m$ to describe these 8 first-class constraints where $A=1$ to 8 is an SO(8) chiral spinor index, $\dot{A}=1$ to 8 is an SO(8) antichiral spinor index, $J=1$ to 8 is an SO(8) vector index, and $\Gamma^\pm \equiv \Gamma^0 \pm \Gamma^9$. Note that $\{G^A, G^B\} = -\frac{1}{2}\delta^{AB}P^+P^mP_m$.

Assuming that P^+ is nonzero, one can use G^A to gauge-fix $(\Gamma^+ f)^A = 0$ and can use the $P^2=0$ constraint to gauge-fix $e=0$. In this gauge, the BRST operator is

$$Q = -\frac{1}{2}cP^mP_m + \gamma_A G^A - \frac{1}{2}P^+\gamma_A\gamma_A b \tag{4.2}$$

with the action

$$S = \int d\tau [P_m \partial_\tau x^m + p_\alpha \partial_\tau \theta^\alpha + b\partial_\tau c + \beta^A \partial_\tau \gamma_A + f^{\dot{A}} d_{\dot{A}}]$$

where (β^A, γ_A) are bosonic ghosts coming from the gauge-fixing of f^A=0, and $f^{\dot{A}} d_{\dot{A}} = f^\alpha(\Gamma^- \Gamma^+ d)_\alpha$ describe the remaining second-class constraints.

To complete the BRST quantization, one needs to express the 8 second-class constraints $d_{\dot{A}} = 0$ in terms of 4 first-class constraints. This is done by first splitting the eight components of γ^A as

$$\gamma_A = \delta_A^+ \widetilde{\gamma}_+ + (P^+)^{-1} \lambda_A \tag{4.3}$$

where λ_A is a null SO(8) spinor satisfying $\lambda_A \lambda_A = 0$. More explicitly, one decomposes the SO(8) spinor γ_A into U(4) components as $\gamma_A \to (\gamma_+, \gamma_{jk}, \gamma_-)$ where $j = 1$ to 4, and defines

$$\lambda^+ \equiv \lambda_- = P^+ \gamma_-, \qquad \lambda_{jk} = P^+ \gamma_{jk}, \tag{4.4}$$

$$\lambda_+ = -\frac{1}{8} (\lambda^+)^{-1} \epsilon^{jklm} \lambda_{jk} \lambda_{lm}, \tag{4.5}$$

$$\widetilde{\gamma}_+ = \gamma_+ - (P^+)^{-1} \lambda_+. \tag{4.6}$$

In terms of $\widetilde{\gamma}_+$ and λ^A, the BRST operator and action of (4.2) are

$$Q = -\frac{1}{2} c P^m P_m + \widetilde{\gamma}_+ G^+ + (P^+)^{-1} \lambda_A G^A - \widetilde{\gamma}_+ \lambda^+ b \tag{4.7}$$

and

$$S = \int d\tau [P_m \partial_\tau x^m + p_\alpha \partial_\tau \theta^\alpha + b\partial_\tau c + \widetilde{\beta}^+ \partial_\tau \widetilde{\gamma}_+ + w_A \partial_\tau \lambda^A + f^{\dot{A}} d_{\dot{A}}]. \tag{4.8}$$

One then defines the first-class constraints as

$$H^J = \lambda \Gamma^- \Gamma^J d = \lambda^A \sigma^J_{A\dot{A}} d^{\dot{A}} \tag{4.9}$$

where $\sigma^J_{A\dot{A}}$ are the SO(8) Pauli matrices. Note that $\lambda^A \lambda^A = 0$ implies that only 4 of the 8 components of H^J are independent. And since the components of H^J anticommute with each other and with the BRST operator of (4.7), they can be used to replace the 8 second-class constraints $d_{\dot{A}} = 0$.

So one can replace (4.8) with the action

$$S = \int d\tau [P_m \partial_\tau x^m + p_\alpha \partial_\tau \theta^\alpha + b\partial_\tau c + \widetilde{\beta}^+ \partial_\tau \widetilde{\gamma}_+ + w_A \partial_\tau \lambda^A + h_J H^J] \tag{4.10}$$

where only four components of the Lagrange multipliers h^J are nonzero (e.g. choose $h^5 = h^6 = h^7 = h^8 = 0$). Note that the action of (4.8) is recovered if one uses the first-class constraints of (4.9) to gauge $d_{\dot{a}} = 0$, which produces no new propagating ghosts.[4] However, one can also use (4.9) to gauge $h^J = 0$, in which case the resulting BRST operator and action are

$$Q = -\frac{1}{2} c P^m P_m + \widetilde{\gamma}_+ G^+ + (P^+)^{-1} \lambda_A G^A - \widetilde{\gamma}_+ \lambda^+ b + \gamma_J H^J, \tag{4.11}$$

$$S = \int d\tau [P_m \partial_\tau x^m + p_\alpha \partial_\tau \theta^\alpha + b\partial_\tau c + \widetilde{\beta}^+ \partial_\tau \widetilde{\gamma}_+ + w_A \partial_\tau \lambda^A + \beta_J \partial_\tau \gamma^J] \tag{4.12}$$

[4] I thank Yuri Aisaka for discussions on this point.

where γ_J are bosonic ghosts with only four nonzero components. Finally, since $G^A = P^+ d^A + P^J \sigma_J^{A\dot{A}} d_{\dot{A}}$, the BRST operator of (4.11) is equal to the superparticle BRST operator in the pure spinor formalism,

$$Q = -\frac{1}{2} c P^m P_m + \widetilde{\gamma}_+ G^+ + \lambda^\alpha d_\alpha - \widetilde{\gamma}_+ \lambda^+ b \tag{4.13}$$

where $\lambda^\alpha d_\alpha = \lambda^A d_A + \lambda^{\dot{A}} d_{\dot{A}}$ and $\lambda_{\dot{A}}$ is defined as

$$\lambda_{\dot{A}} \equiv (\gamma_J + P_J) \sigma^J_{A\dot{A}} \lambda^A \tag{4.14}$$

which has four independent components. Note that $\lambda^{\dot{A}} \lambda^{\dot{A}} = \lambda^A \lambda^A = \lambda^A \sigma^J_{A\dot{A}} \lambda^{\dot{A}} = 0$, so λ^α is a pure spinor with 11 independent components.

4.2 Green-Schwarz superstring

To extend these results from the Brink-Schwarz superparticle to the Green-Schwarz superstring, first write the Green-Schwarz action in first-order form as [19, 10]

$$S = \int d^2 z \Big[\frac{1}{2} \partial x^m \overline{\partial} x_m + p_\alpha \overline{\partial} \theta^\alpha + \widehat{p}_{\widehat{\alpha}} \partial \widehat{\theta}^{\widehat{\alpha}} \tag{4.15}$$

$$+ f^\alpha d_\alpha + \widehat{f}^{\widehat{\alpha}} \widehat{d}_{\widehat{\alpha}} - e \Big(\frac{1}{2} \partial x^m \partial x_m + p_\alpha \partial \theta^\alpha \Big) - \widehat{e} \Big(\frac{1}{2} \overline{\partial} x^m \overline{\partial} x_m + \widehat{p}_{\widehat{\alpha}} \overline{\partial} \widehat{\theta}^{\widehat{\alpha}} \Big) \Big] \tag{4.16}$$

where

$$\begin{aligned} d_\alpha &= p_\alpha - \frac{1}{2} \partial x^m (\Gamma_m \theta)_\alpha - \frac{1}{8} (\theta \Gamma^m \partial \theta)(\Gamma_m \theta)_\alpha, \\ \widehat{d}_{\widehat{\alpha}} &= \widehat{p}_{\widehat{\alpha}} - \frac{1}{2} \overline{\partial} x^m (\Gamma_m \widehat{\theta})_{\widehat{\alpha}} - \frac{1}{8} (\widehat{\theta} \Gamma^m \overline{\partial} \widehat{\theta})(\Gamma_m \widehat{\theta})_{\widehat{\alpha}}, \end{aligned} \tag{4.17}$$

f^α and $\widehat{f}^{\widehat{\alpha}}$ are fermionic Lagrange multipliers, e and $\widehat{e}$ are the off-diagonal components of the worldsheet metric, and $(\alpha, \widehat{\alpha})$ are spinor indices of the opposite/same chirality for the Type IIA/IIB superstring. In the following discussion, only the unhatted variables will be gauge-fixed, however, one can gauge-fix the hatted variables in an identical manner.

As in the superparticle, d_α contains 8 first-class and 8 second-class constraints. The first-class constraints are generated by 8 of the 16 components of $\Pi_m (\Gamma^m d)^\alpha$ where $\Pi^m = \partial x^m + \frac{1}{2} \theta \Gamma^m \partial \theta$ is the supersymmetric momentum. Choosing

$$\widetilde{G}^A = \frac{1}{2} (\Gamma^+ \Gamma^m d)^A \Pi_m \tag{4.18}$$

to describe the 8 independent first-class constraints, one follows the same steps as in the superparticle and gauge-fixes $(\Gamma^+ f)^A = e = 0$.

As shown in [10], $\widetilde{G}^A$ satisfies the Poisson brackets

$$\begin{aligned} \{\widetilde{G}^A(\sigma_1), \widetilde{G}^B(\sigma_2)\} = \delta(\sigma_1 - \sigma_2) \Big[& \partial \theta^{(A} \widetilde{G}^{B)} + \delta^{AB} \partial \theta^C \widetilde{G}^C \\ & + \delta^{AB} \Pi^+ \Big(-\frac{1}{2} \Pi^m \Pi_m - d_\alpha \partial \theta^\alpha \Big) + \frac{1}{2} \delta^{AB} d_{\dot{A}} \partial d_{\dot{A}} \Big]. \end{aligned} \tag{4.19}$$

So the BRST operator and action after gauge-fixing are

$$Q = \int dz \Big[c\Big(-\frac{1}{2}\Pi^m\Pi_m - d_\alpha\partial\theta^\alpha + \beta\partial\gamma + \partial(\beta\gamma) - b\partial c\Big) - \frac{1}{2}\Pi^+\gamma^A\gamma^A b + \gamma^A\widetilde{G}^A \tag{4.20}$$
$$-(\gamma_A\partial\theta^A)(\gamma_B\beta^B) - \frac{1}{2}(\gamma_A\gamma^A)(\beta_B\partial\theta^B)\Big],$$

$$S = \int d^2z \Big[\frac{1}{2}\partial x^m\overline{\partial}x_m + p_\alpha\overline{\partial}\theta^\alpha + b\overline{\partial}c + \beta_A\overline{\partial}\gamma^A + f^{\dot{A}}d_{\dot{A}}\Big], \tag{4.21}$$

where the last term in (4.19) can be ignored since it is quadratic in the second-class constraints.

As in the superparticle, the 8 second-class constraints $d_{\dot{A}} = 0$ can be converted into 4 first-class constraints by writing $\gamma_A = \delta^+_A\widetilde{\gamma}_+ + (\Pi^+)^{-1}\lambda_A$ where $\lambda_A\lambda_A = 0$, and defining the first-class constraints as

$$H^J = \lambda^A\sigma^J_{A\dot{A}}d^{\dot{A}} \tag{4.22}$$

which has only four independent components.

It is not difficult to verify that H^J anticommutes with H^K and satisfies

$$\{Q, H^J\} = \partial(cH^J) + \widetilde{\gamma}_+H^K(\Gamma^K\Gamma^J\partial\theta)^+ + (\Pi^+)^{-1}(\lambda^A\partial\theta^A)H^J, \tag{4.23}$$

so H^J describe first-class constraints which can replace the 8 second-class constraints $d_{\dot{A}}$. After gauge-fixing the Lagrange multiplier $h^J = 0$ as in the superparticle, the BRST operator of (4.21) becomes

$$Q = \int dz \Big[c\Big(-\frac{1}{2}\Pi^m\Pi_m - d_\alpha\partial\theta^\alpha + w_A\partial\lambda^A + \beta_J\partial\lambda^J + \widetilde{\beta}^+\partial\widetilde{\gamma}_+ + \partial(\widetilde{\beta}^+\widetilde{\gamma}_+) - b\partial c\Big)$$
$$-\widetilde{\gamma}_+\lambda^+b + \widetilde{\gamma}_+\widetilde{G}^+ + \lambda_A\widetilde{G}^A + \gamma_J H^J + \ldots\Big] \tag{4.24}$$

where $\ldots$ involves ghost-ghost-antighost terms multiplied by components of $\partial\theta^\alpha$. Finally, after defining $\lambda_{\dot{A}} = (\gamma_J + \Pi_J)\sigma^J_{A\dot{A}}\lambda^A$ as in the superparticle, one obtains the pure spinor BRST operator of (2.28)

$$Q = \int dz(c\widetilde{T} + \widetilde{\gamma}_+G^+ + \lambda^\alpha d_\alpha - \widetilde{\gamma}_+\lambda^+b + bc\partial c) \tag{4.25}$$

where $\widetilde{T} = -\frac{1}{2}\partial x^m\partial x_m - p_\alpha\partial\theta^\alpha + \widetilde{T}_{\rm pure}$, and $G^+ = \widetilde{G}^+ - \frac{1}{4}N_{mn}(\Gamma^{mn}\partial\theta)^+ - \frac{1}{4}J\partial\theta^+ - \frac{1}{4}\partial^2\theta^+$ is defined as in (2.19). Although $G^+ - \widetilde{G}^+$ can be determined by computing the ghost-ghost-antighost terms in (4.24), $G^+ - \widetilde{G}^+$ can also be indirectly determined by requiring the nilpotence of Q.

5. Mapping RNS into the pure spinor formalism

In this section, the RNS BRST operator will be mapped into the pure spinor BRST operator by a field redefinition which maps the RNS variables into Green-Schwarz-Siegel variables. For states in the Neveu-Schwarz $GSO(+)$ sector, the RNS and pure spinor vertex operators in the zero picture will then be mapped into each other. However, since Ramond states in the RNS formalism and $GSO(-)$ states in the pure spinor formalism do not have vertex operators in the zero picture, there is no obvious way to map their vertex operators into each other.

5.1 Twisting the RNS fields

The first step in performing the map from the RNS BRST operator to the pure spinor BRST operator is to twist the ten spin-half RNS fermions ψ_m into five spin-zero fermions $\widetilde{\psi}_m$ and five spin-one fermions $\widetilde{\widetilde{\psi}}_m$ as [6]

$$\psi^m = \frac{1}{\gamma}\widetilde{\psi}_n \frac{(\lambda\Gamma^m\Gamma^n)^+}{2\lambda^+} + \gamma\widetilde{\widetilde{\psi}}_n \frac{(\lambda\Gamma^n\Gamma^m)^+}{2\lambda^+} \tag{5.1}$$

where γ is the RNS bosonic ghost of $-\frac{1}{2}$ conformal weight, $(\lambda\Gamma^m\Gamma^n)^+$ is the component of $(\lambda\Gamma^m\Gamma^n)^\alpha$ with $\frac{5}{2}$ U(1) charge, and λ^α is a pure spinor which parameterizes the SO(10)/U(5) different choices for twisting. Note that only 5 independent components of $\widetilde{\psi}_m$ and $\widetilde{\widetilde{\psi}}_m$ contribute to (5.1), and (5.1) can be inverted to imply that

$$\widetilde{\psi}_n(\lambda\Gamma^m\Gamma^n)^+ = \gamma\psi_n(\lambda\Gamma^m\Gamma^n)^+, \quad \widetilde{\widetilde{\psi}}_n(\lambda\Gamma^n\Gamma^m)^+ = \frac{1}{\gamma}\psi_n(\lambda\Gamma^n\Gamma^m)^+. \tag{5.2}$$

Since the spin $\frac{3}{2}$ bosonic antighost β has non-trivial OPE's with $\widetilde{\psi}_m$ and $\widetilde{\widetilde{\psi}}_m$, it is convenient to define new fields $\widetilde{\beta} = \partial\widetilde{\xi}e^{-\widetilde{\phi}}$ and $\widetilde{\gamma} = \widetilde{\eta}e^{\widetilde{\phi}}$ where

$$\widetilde{\eta} = e^{-\frac{1}{2}\phi}\lambda^\alpha\Sigma_\alpha, \qquad \widetilde{\xi} = e^{\frac{1}{2}\phi}(\lambda^+)^{-1}\overline{\Sigma}^+, \tag{5.3}$$

$$e^{\widetilde{\phi}} = \eta\partial\eta e^{\frac{5}{2}\phi}(\lambda^+)^{-1}\overline{\Sigma}^+, \qquad e^{-\widetilde{\phi}} = \xi\partial\xi e^{-\frac{5}{2}\phi}\lambda^\alpha\Sigma_\alpha, \tag{5.4}$$

and Σ_α and $\overline{\Sigma}^\alpha$ are anti-Weyl and Weyl spin fields of $\frac{5}{8}$ conformal weight which are constructed in the usual manner from the ψ^m variables. The definitions of (5.3) are uniquely determined by the requirements that $[\widetilde{\eta}, \widetilde{\xi}, e^{\widetilde{\phi}}, e^{-\widetilde{\phi}}]$ have the same OPE's as $[\eta, \xi, e^\phi, e^{-\phi}]$ with each other, that $[\widetilde{\eta}, \widetilde{\xi}, e^{\widetilde{\phi}}, e^{-\widetilde{\phi}}]$ have no poles with $\widetilde{\psi}_m$ and $\widetilde{\widetilde{\psi}}_m$, and that $\widetilde{\eta}$ has $+1$ conformal weight. Note that

$$\widetilde{\gamma} = \widetilde{\eta}e^{\widetilde{\phi}} = \eta\partial\eta e^{2\phi} = \gamma^2 \tag{5.5}$$

carries spin -1 and $\widetilde{\beta}$ carries spin 2. So the twisting of (5.1) and (5.3) shifts the central charge contribution of the (β, γ) ghosts from 11 to 26, which cancels the shift from 5 to -10 in the central charge contribution of the twisted $\widetilde{\psi}$'s.

If λ^α is treated as a worldsheet field, one needs to introduce a fermionic superpartner for λ^α and add a topological term to the RNS BRST operator so that these new fields do not contribute to the cohomology. The fermionic superpartner to λ^α will be called $\widetilde{\theta}^\alpha$ for reasons that will become clear, and will be defined to transform under BRST as

$$Q\widetilde{\theta}^\alpha = \lambda^\alpha, \quad Q\lambda^\alpha = 0. \tag{5.6}$$

Furthermore, because of the pure spinor constraint $\lambda\Gamma^m\lambda = 0$, $\widetilde{\theta}^\alpha$ will be required to satisfy the fermionic constraint

$$\widetilde{\theta}^\alpha\Gamma^m_{\alpha\beta}\lambda^\beta = 0. \tag{5.7}$$

It is easy to verify that the constraint of (5.7) eliminates five components of $\widetilde{\theta}^\alpha$, so that $\widetilde{\theta}^\alpha$ and λ^α each have eleven independent components.

To generate the BRST transformation of (5.6), one should add $\int dz \lambda^\alpha \widetilde{p}_\alpha$ to the RNS BRST operator so that

$$Q = \int dz [cT_{\text{RNS}} - \gamma \partial x^m \psi_m + bc\partial c + \gamma^2 b + \lambda^\alpha \widetilde{p}_\alpha] \tag{5.8}$$

where w_α is the conjugate momenta to λ^α and $\widetilde{p}_\alpha$ is the conjugate momenta to $\widetilde{\theta}^\alpha$. Because of the constraints $\lambda\Gamma^m\lambda = \lambda\Gamma^m\widetilde{\theta} = 0$, w_α and $\widetilde{p}_\alpha$ are defined up to the gauge transformations

$$\delta w_\alpha = \rho^m (\Gamma_m \lambda)_\alpha + \Omega^m (\Gamma_m \widetilde{\theta})_\alpha, \quad \delta \widetilde{p}_\alpha = \Omega^m (\Gamma_m \lambda)_\alpha, \tag{5.9}$$

where ρ^m and Ω^m are arbitrary gauge parameters. So five of the sixteen parameters of each of these conjugate momenta can be gauged away.

To construct super-Poincaré covariant Green-Schwarz-Siegel variables out of the RNS variables, one can now combine the eleven components of $\widetilde{\theta}^\alpha$ and $\widetilde{p}_\alpha$ with the five spin-zero and spin-one components of $\widetilde{\psi}_m$ and $\widetilde{\widetilde{\psi}}_m$ to define the unconstrained sixteen-component spinors

$$\theta^\alpha = \widetilde{\theta}^\alpha + \widetilde{\psi}_m \frac{(\Gamma^m)^{\alpha +}}{2\lambda^+}, \quad p_\alpha = \widetilde{p}_\alpha + \widetilde{\widetilde{\psi}}_m (\Gamma^m \lambda)_\alpha. \tag{5.10}$$

Note that (5.10) implies that ψ^m can be expressed in terms of p_α and θ^α as

$$\psi^m = \gamma \frac{(\Gamma^m p)^+}{2\lambda^+} + \frac{1}{\gamma}(\lambda \Gamma^m \theta), \tag{5.11}$$

and the OPE $\psi^m(y)\psi^n(z) \to (y-z)^{-1}\eta^{mn}$ implies that $p_\alpha(y)\theta^\beta(z) \to (y-z)^{-1}\delta_\alpha^\beta$.

When expressed in terms of p_α and θ^α,

$$\gamma \partial x^m \psi_m = (\lambda \Gamma_m \theta)\partial x^m + \widetilde{\gamma} \frac{(\Gamma_m p)^+}{2\lambda^+} \partial x^m. \tag{5.12}$$

And $\lambda\Gamma^m\lambda = 0$ implies that $\lambda^\alpha p_\alpha = \lambda^\alpha \widetilde{p}_\alpha$. So the BRST operator of (5.8) can be written as

$$Q = \int dz \left[cT_{\text{RNS}} + bc\partial c + \lambda^\alpha (p_\alpha - (\Gamma^m \theta)_\alpha \partial x_m) + \widetilde{\gamma} \left(b - \frac{(\Gamma^m p)^+}{2\lambda^+} \partial x_m \right) \right] \tag{5.13}$$

where

$$T_{\text{RNS}} = -\frac{1}{2}\partial x^m \partial x_m + (\widetilde{\gamma})^{-1}(\theta \Gamma^m \lambda)(\theta \Gamma_m \partial \lambda) - \frac{(\Gamma^m p)^+}{2\lambda^+}\partial(\theta \Gamma_m \lambda) + \widetilde{\beta}\partial\widetilde{\gamma} + \partial(\widetilde{\beta}\widetilde{\gamma}). \tag{5.14}$$

Finally, to put (5.14) into the standard form for a stress tensor and to covariantize $(\Gamma^m p)^+$ into $(\Gamma^m d)^+$, one performs the similarity transformation $Q \to e^U e^S Q e^{-S} e^{-U}$ where (up to possible errors in the coefficients)

$$\begin{aligned} S &= \int dz\ c \left[(\widetilde{\gamma})^{-1} (\lambda \Gamma^m \theta)(\theta \Gamma_m \partial \theta) + (\lambda^+)^{-1} \left(-\frac{1}{8} (\Gamma^{mn} \partial \theta)^+ (w \Gamma_{mn} \lambda) - \frac{1}{4} \partial \theta^+ (w \lambda) \right) \right], \\ U &= \int dz\ (16\lambda^+)^{-1} (\Gamma_{mn} \lambda)^+ \partial x_p (\theta \Gamma^{mnp} \theta). \end{aligned} \tag{5.15}$$

After performing this similarity transformation,

$$Q = \int dz \left[c\widetilde{T} + bc\partial c + \lambda^\alpha d_\alpha + \widetilde{\gamma}\left(b - \frac{G^+}{\lambda^+} \right) \right] = e^R \left[\int dz\ (\lambda^\alpha d_\alpha + \widetilde{\gamma} b) \right] e^{-R} \tag{5.16}$$

where $\widetilde{T}$, G^+, d_α and R are defined as in section 2. So the RNS BRST operator has been mapped into the pure spinor BRST operator of (2.22).

5.2 Neveu-Schwarz $GSO(+)$ vertex operators

In this subsection, the Neveu-Schwarz $GSO(+)$ vertex operators in the zero picture in the RNS formalism will be mapped into the corresponding pure spinor vertex operators. However, since Ramond vertex operators in the RNS formalism and $GSO(-)$ vertex operators in the pure spinor formalism cannot be written in the zero picture, there is no obvious way to relate the vertex operators for these states in the two formalisms. Note that the map of (5.3) acts in a simple manner on operators in the zero picture, i.e. operators which can be expressed directly in terms of γ and $\widetilde{\gamma}$. However, the map acts in a complicated manner on operators in nonzero picture which contain explicit ϕ or $\widetilde{\phi}$ dependence.

In the zero picture, unintegrated Neveu-Schwarz vertex operators in the RNS formalism have the form

$$V_{\text{RNS}} = \gamma W + cG_{-\frac{1}{2}} W \tag{5.17}$$

where W is an $N = 1$ superconformal primary of weight $\frac{1}{2}$ constructed from (x^m, ψ^m) and $G_{-\frac{1}{2}} W$ is the single pole of $\psi_m \partial x^m$ with W. After performing the field redefinition of (5.11), V_{RNS} is expressed in terms of the variables $[x^m, (\lambda\gamma^m\theta),\ (\lambda^+)^{-1}(\gamma^m p)^+, \widetilde{\gamma}, c]$. And if the state is $GSO(+)$, this operator contains integer powers of $\widetilde{\gamma}$.

To map V_{RNS} to a pure spinor vertex operator, one needs to perform the similarity transformation $V = e^{-R} e^U e^S V_{\text{RNS}} e^{-S} e^{-U} e^R$ where R, S and U are defined in (2.21) and (5.15). Since

$$e^{-R} e^U e^S \left(Q_{\text{RNS}} + \int dz\ \lambda^\alpha p_\alpha \right) e^{-S} e^{-U} e^R = \int dz (\lambda^\alpha d_\alpha + \widetilde{\gamma} b), \tag{5.18}$$

V is in the pure spinor cohomology. But before claiming that V is a pure spinor vertex operator, one needs to ensure it is independent of inverse powers of λ^+ and $\widetilde{\gamma}$. One can show that any dependence on such inverse powers can be removed by adding a suitable BRST-trivial operator, however, the form of this BRST-trivial operator may be complicated to construct.

A more direct way to map the Neveu-Schwarz $GSO(+)$ vertex operator of (5.17) into the corresponding pure spinor vertex operator is to write the $N = 1$ superconformal primary W of (5.17) in the form

$$W = \psi^m f_m(x^n, M_{pq}) \tag{5.19}$$

where $M_{pq} = \psi_p \psi_q$ is the contribution of ψ^m to the RNS Lorentz current, and $f_m(x^n, M_{pq})$ is a function of x^n and M_{pq} and their worldsheet derivatives. Since $GSO(+)$ superconformal primaries have an odd number of ψ fields, it is always possible to write W in the form of (5.19) for some choice of $f_m(x^n, M_{pq})$.

The corresponding pure spinor vertex operator will then be defined as

$$V = (\lambda\gamma^m\theta) f_m(x, M) + \sum_{n=1}^{\infty} V_{2n+1} \tag{5.20}$$

where

$$M_{pq} = N_{pq} + \frac{1}{2}(p\Gamma_{pq}\theta) \tag{5.21}$$

and the terms in V_{2n+1} contain $(2n+1)$ more θ's than p's. Note that (2.10) implies that M_{pq} of (5.21) has the same OPE's as $M_{pq} = \psi_p\psi_q$. To determine the terms in V_{2n+1}, use $\{\int dz\ \lambda^\alpha d_\alpha, V\} = 0$ and $\{\int dz\ \lambda^\alpha p_\alpha, (\lambda\gamma^m\theta) f_m(x, M)\} = 0$ to imply that

$$\begin{aligned}\left\{\int dz\ \lambda^\alpha p_\alpha,\ V_{2n+1}\right\} = &\left\{\frac{1}{2}\int dz\ \lambda^\alpha \partial x^m (\Gamma_m\theta)_\alpha,\ V_{2n-1}\right\} + \\ &\left\{\frac{1}{8}\int dz\ \lambda^\alpha (\theta\Gamma^m\partial\theta)(\Gamma_m\theta)_\alpha,\ V_{2n-3}\right\}\end{aligned} \tag{5.22}$$

where $V_1 = (\lambda\gamma^m\theta) f_m(x, M)$ and $V_m = 0$ for $m < 0$.

Finding solutions to (5.22) for V_{2n+1} would always be possible if $\int dz\ \lambda^\alpha p_\alpha$ had trivial cohomology at $+2$ ghost number. Although in fact there are non-trivial elements at $+2$ ghost number in the cohomology of $\int dz\ \lambda^\alpha p_\alpha$ (e.g. the state $(\lambda\Gamma^m\theta)(\lambda\Gamma^n\theta)(\theta\Gamma_{mnp}\theta)$), it seems reasonable to conjecture that when $W = \psi^m f_m(x, M)$ is an $N = 1$ superconformal primary, these non-trivial elements are not an obstacle to finding solutions for V_{2n+1} which satisfy (5.22). So assuming this conjecture concerning $\int dz\ \lambda^\alpha p_\alpha$ cohomology at $+2$ ghost number, there is a simple map from unintegrated Neveu-Schwarz $GSO(+)$ vertex operators in the RNS formalism to unintegrated vertex operators in the pure spinor formalism.

One can similarly map integrated Neveu-Schwarz $GSO(+)$ vertex operators at zero picture in the RNS formalism into the corresponding pure spinor vertex operators. If

$$\int dz\ U_{\text{RNS}} = \int dz\ f(x, M) \tag{5.23}$$

is the integrated vertex operator in the RNS formalism where $M_{mn} = \psi_m\psi_n$, then

$$\int dz\ U = \int dz\ \left[f(x, M) + \sum_{n=1}^{\infty} U_{2n}\right] \tag{5.24}$$

is the integrated vertex operator in the pure spinor formalism where $M_{mn} = N_{mn} + \frac{1}{2}(p\Gamma_{mn}\theta)$ and U_{2n} contains $2n$ more θ's than p's. In this case, finding solutions to U_{2n} is related to the cohomology of $\int dz\ \lambda^\alpha p_\alpha$ at $+1$ ghost number. When $\int dz\ U_{\text{RNS}}$ is $N = 1$ superconformally invariant, one expects that non-trivial elements in this cohomology do not provide obstacles to solving for U_{2n}.

The maps of (5.20) and (5.24) can easily be verified for the massless gluon vertex operator where $W = \psi^m a_m(x)$ and $U_{\text{RNS}} = \partial x^m a_m(x) + M^{mn}\partial_m a_n(x)$. And since any massive Neveu-Schwarz $GSO(+)$ vertex operator can be obtained from the OPE's of gluon vertex operators, this map is indirectly verified also for massive states. Furthermore, since there are no terms in these vertex operators with more p's than θ's, most of the terms

V_{2n+1} and U_{2n} in the pure spinor vertex operators will not contribute. So using arguments similar to [20] one can verify that tree amplitudes involving Neveu-Schwarz $GSO(+)$ states coincide in the two formalisms. However, since loop amplitudes involve intermediate states in the Ramond $GSO(+)$ sector, it is not surprising that it is difficult to prove equivalence of the RNS and pure spinor amplitude prescriptions for loop amplitudes.

6. Inverse map for $N = 0 \to N = 1$ embedding

After twisting the ten RNS spin-half fields into five spin-zero and five spin-one fields, the RNS superstring was mapped in the previous section into the pure spinor formalism. Since the pure spinor formalism can be interpreted as an $N = 2$ topological string [3], which is a natural generalization of bosonic strings, the map takes a critical $N = 1$ string into a type of $N = 0$ string.

As shown with Vafa [7], any critical $N = 0$ string can be embedded into a critical $N = 1$ string by twisting the (b, c) ghosts from spin $(2, -1)$ to spin $(\frac{3}{2}, -\frac{1}{2})$ and defining the $N = 1$ superconformal generator as $G = b + j_{\text{BRST}}$. In this section, it will be shown that if one starts with the $N = 1$ string corresponding to this $N = 0 \to N = 1$ embedding of the bosonic string and performs the map of the previous section, one recovers the original $N = 0$ bosonic string.

So the map of the previous section from the RNS to the pure spinor formalism can be interpreted as an inverse map for the $N = 0 \to N = 1$ embedding of [7]. This interpretation suggests there may be generalizations of the pure spinor formalism which would arise by applying the inverse map to other types of critical $N = 1$ superconformal field theories.

6.1 Review of $N = 0 \to N = 1$ embedding

In this subsection, the map of [7] from a critical $N = 0$ string to a critical $N = 1$ string will be reviewed. Suppose one starts with a $c = 26$ matter system with stress tensor T_m. Then the standard quantization as a critical $N = 0$ string is to introduce (b, c) ghosts of conformal weight $(2, -1)$ and define physical states using the $N = 0$ BRST operator

$$Q_{N=0} = \int dz[cT_m + bc\partial c]. \tag{6.1}$$

However, the same matter system can also be quantized as a critical $N = 1$ string by adding a set of (b_1, c_1) matter fields of conformal weight $(\frac{3}{2}, -\frac{1}{2})$ so the combined system has central charge 15. One then defines a set of critical $N = 1$ superconformal generators as

$$T_{N=1} = T_m - b_1\partial c_1 - \frac{1}{2}\partial(b_1c_1) + \frac{1}{2}\partial^2(c_1\partial c_1), \tag{6.2}$$

$$G_{N=1} = c_1(T_m + \partial c_1 b_1) + \frac{5}{2}\partial^2 c_1 + b_1. \tag{6.3}$$

Note that $G_{N=1} = j_{\text{BRST}} + b_1$ where, up to a total derivative, j_{BRST} is the BRST current of (6.1) with (b, c) replaced by (b_1, c_1).

One can now perform the standard $N = 1$ quantization by introducing fermionic (b, c) ghosts of conformal weight $(2, -1)$ and bosonic (β, γ) ghosts of conformal weight $(\frac{3}{2}, -\frac{1}{2})$, and defining physical states using the $N = 1$ BRST operator

$$Q_{N=1} = \int dz \left[cT_{N=1} + \gamma G_{N=1} - \gamma^2 b + bc\partial c + c\left(\beta\partial\gamma + \frac{1}{2}\partial(\beta\gamma)\right)\right]. \tag{6.4}$$

Equivalence of the cohomologies of $Q_{N=0}$ of (6.1) and $Q_{N=1}$ of (6.4) was proven in [21] by writing

$$Q_{N=1} = e^U \left[\int dz(cT_m + bc\partial c + \gamma b_1)\right] e^{-U} \tag{6.5}$$

where

$$U = \int dz c_1 \left(\frac{1}{2}\gamma b - \frac{3}{2}\partial c\beta - c\partial\beta + \frac{1}{2}\partial c_1 cb - \frac{1}{4}\beta\gamma\partial c_1\right).$$

The topological term γb_1 in (6.5) implies that the $N = 1$ cohomology is independent of (β, γ) and (b_1, c_1), so the $Q_{N=1}$ cohomology coincides with the $Q_{N=0}$ cohomology. Furthermore, it was shown in [7] that the $N = 1$ amplitude prescription coincides with the $N = 0$ amplitude prescription where the functional integral over the bosonic (β, γ) fields cancels the functional integral over the fermionic (b_1, c_1) fields.

6.2 Inverse map for bosonic string

In this subsection, it will be shown that if one starts with the $N = 1$ string coming from the $N = 0 \to N = 1$ embedding of the bosonic string and performs similar steps as in the map from the RNS to the pure spinor formalism, one ends up with the original $N = 0$ description of the bosonic string.

The first step is to twist the (b_1, c_1) matter fields from spin $(\frac{3}{2}, -\frac{1}{2})$ to $(2, -1)$ by defining [6]

$$\widetilde{b}_1 = \frac{1}{\gamma} b_1, \quad \widetilde{c}_1 = \gamma c_1 \tag{6.6}$$

as in the twisting of the ψ^m matter fields in the RNS formalism. Since β has non-trivial OPE's with $\widetilde{b}_1$ and $\widetilde{c}_1$, it is convenient to define new fields $\widetilde{\beta} = \partial\widetilde{\xi} e^{-\widetilde{\phi}}$ and $\widetilde{\gamma} = \widetilde{\eta} e^{\widetilde{\phi}}$ where

$$\widetilde{\eta} = \eta e^{\frac{1}{2}(\phi - i\sigma)}, \qquad \widetilde{\xi} = \xi e^{\frac{1}{2}(-\phi + i\sigma)}, \tag{6.7}$$
$$e^{\widetilde{\phi}} = \eta e^{\frac{1}{2}(3\phi + i\sigma)}, \qquad e^{-\widetilde{\phi}} = \xi e^{\frac{1}{2}(-3\phi - i\sigma)},$$

and $c_1 = e^{i\sigma}$ and $b_1 = e^{-i\sigma}$. The definitions of (6.7) are uniquely determined by the requirements that $[\widetilde{\eta}, \widetilde{\xi}, e^{\widetilde{\phi}}, e^{-\widetilde{\phi}}]$ have the same OPE's as $[\eta, \xi, e^{\phi}, e^{-\phi}]$ with each other, that $[\widetilde{\eta}, \widetilde{\xi}, e^{\widetilde{\phi}}, e^{-\widetilde{\phi}}]$ have no poles with $\widetilde{c}_1 = \eta e^{\phi + i\sigma}$ and $\widetilde{b}_1 = \xi e^{-\phi - i\sigma}$, and that $\widetilde{\eta}$ has $+1$ conformal weight. One can easily verify from (6.7) that

$$\widetilde{\gamma} = \eta\partial\eta e^{2\phi} = \gamma^2 \tag{6.8}$$

carries spin -1 and $\widetilde{\beta}$ carries spin 2. So the twisting of (6.6) and (6.7) has shifted the spins of both (b_1, c_1) and (β, γ) from $(\frac{3}{2}, -\frac{1}{2})$ to $(2, -1)$, and their contributions to the central charge continue to cancel each other.

When written in terms of $(\widetilde{b}_1, \widetilde{c}_1)$ and $(\widetilde{\beta}, \widetilde{\gamma})$, $Q_{N=1}$ of (6.4) becomes

$$Q_{N=1} = \int dz \left[cT + \widetilde{c}_1(T_m + \partial\widetilde{c}_1\widetilde{b}_1) + \widetilde{\gamma}(b - \widetilde{b}_1) - \frac{1}{2}\widetilde{\gamma}\partial^2\left(\frac{\widetilde{c}_1}{\widetilde{\gamma}}\right) + bc\partial c \right] \tag{6.9}$$

where

$$T = T_m + \frac{1}{2}\partial^2(\frac{\widetilde{c}_1\partial\widetilde{c}_1}{\widetilde{\gamma}}) - \widetilde{b}_1\partial\widetilde{c}_1 - \partial(\widetilde{b}_1\widetilde{c}_1) + \widetilde{\beta}\partial\widetilde{\gamma} + \partial(\widetilde{\beta}\widetilde{\gamma}). \tag{6.10}$$

To put T into the standard form for a stress tensor, one can perform the similarity transformation $Q \to e^S Q e^{-S}$ where $S = -\frac{1}{2}(\widetilde{\gamma})^{-1}\widetilde{c}_1\partial^2\widetilde{c}_1$ which transforms Q into

$$\begin{aligned} Q &= \int dz[c(T_m - \widetilde{b}_1\partial\widetilde{c}_1 - \partial(\widetilde{b}_1\widetilde{c}_1) + \widetilde{\beta}\partial\widetilde{\gamma} + \partial(\widetilde{\beta}\widetilde{\gamma})) + \widetilde{c}_1(T_m - \widetilde{b}_1\partial\widetilde{c}_1) + \widetilde{\gamma}(b - \widetilde{b}_1) + bc\partial c] \\ &= e^R \int dz[\widetilde{c}_1(T_m + \partial\widetilde{c}_1\widetilde{b}_1) + \widetilde{\gamma}b]e^{-R} = e^R\left(Q'_{N=0} + \int dz\ \widetilde{\gamma}b\right)e^{-R}, \end{aligned} \tag{6.11}$$

where $R = \int dz\ c(\widetilde{b}_1 + \partial c\widetilde{\beta})$ and $Q'_{N=0}$ is the $N = 0$ BRST operator of (6.1) with (b, c) replaced by $(\widetilde{b}_1, \widetilde{c}_1)$.

Since $\widetilde{\gamma}b$ is a topological term, the twisted $(\widetilde{\beta}, \widetilde{\gamma})$ ghosts will now cancel out the contribution of the (b, c) ghosts instead of the (b_1, c_1) matter fields. The remaining fields include the $c = 26$ matter fields and the $(\widetilde{b}_1, \widetilde{c}_1)$ matter fields of $(2, -1)$ conformal weight, which are treated like (b, c) ghosts in the standard $N = 0$ description. So this inverse map takes the $N = 1$ description of the bosonic string into the $N = 0$ description.

7. Comparison with other approaches

In this paper, many mysterious features of the pure spinor formalism were explained by adding a pair of non-minimal fields and performing a similarity transformation which allows the pure spinor BRST operator to be expressed in a conventional-looking form. Although this approach is the first one that has succeeded in describing the $GSO(-)$ sector, there have been several previous approaches to "explaining" the pure spinor formalism and it will be useful to compare this paper with the other approaches.

One approach has been to relax the pure spinor constraint on the ghost variable λ^α and extend the BRST operator to include additional terms which are required for nilpotence [22−24] [25]. Although the conventional-looking BRST operator in this paper also includes additional terms, the extended BRST operators generically require an infinite number of additional terms in order to be nilpotent. It might eventually be possible to relate these extended approaches with the approach of this paper, however, it seems to be much easier to work with the conventional-looking BRST operator which has a finite number of terms. Even though the conventional-looking BRST operator is not manifestly Lorentz invariant, it is easy to show that the resulting scattering amplitudes are Lorentz invariant.

A second approach has been to derive the pure spinor formalism from a semi-light-cone gauge-fixed version of the Green-Schwarz formalism which has double the usual number of θ variables [26, 27]. The resulting equivalence proof with the GS formalism is certainly related to the proof in section 4 of this paper, however, the equivalence proof in this paper is considerably simpler and does not require the choice of semi-light-cone gauge.

JHEP01(2008)065

A third approach has been to interpret the pure spinor formalism as a topological string [3] and to compute scattering amplitudes by coupling to worldsheet topological gravity [8]. Although this approach is probably not useful for comparing with the RNS and GS formalisms, it might eventually be useful for constructing generalizations of the pure spinor formalism, perhaps by looking for other examples of the $N = 1 \to N = 0$ inverse map of section 6.

Finally, a fourth approach has been to relate the pure spinor formalism with the $N = 1 \to N = 2$ embedding of the RNS string [11, 28]. For compactification of the superstring on a Calabi-Yau manifold, this $N = 1 \to N = 2$ embedding is related by a field redefinition to the hybrid formalism [29]. And in ten dimensions, this $N = 1 \to N = 2$ embedding is related by a field redefinition to the GS "twistor string" [28, 30, 31]. If this fourth approach were better understood, it might lead to a proof of equivalence of the RNS and pure spinor multiloop amplitude prescriptions. Furthermore, this approach might allow compactifications of the pure spinor formalism to be related to the hybrid formalism. However, there are some unresolved puzzles concerning this approach.

One puzzle is that the pure spinor formalism appears to be described by a topological $N = 2$ string which has $\hat{c} = 3$, and not by a critical $\hat{c} = 2$ $N = 2$ string which arises from the $N = 1 \to N = 2$ embedding. Note that naive compactification of the pure spinor formalism produces a $\hat{c} = 3$ $N = 2$ theory which, unlike the hybrid formalism, only describes the BPS sector of the compactified superstring [3]. Also, the string field theory action for the pure spinor formalism resembles a Chern-Simons action, as opposed to the Wess-Zumino-Witten-like action [32] which naturally arises from the $N = 1 \to N = 2$ embedding.

Nevertheless, as will be discussed in the following subsection, there is a version of the $N = 1 \to N = 2$ embedding which has many similarities with the fields appearing in the $N = 1 \to N = 0$ embedding and which may eventually be useful for relating the pure spinor and hybrid formalisms. The possibility of using this version of the embedding to relate the pure spinor and hybrid formalisms has been independently observed by Osvaldo Chandía [33].

7.1 $N = 1 \to N = 2$ embedding

The hybrid formalism for the superstring is constructed by first embedding the RNS string into a $\hat{c} = 2$ $N = 2$ string, and then finding a field redefinition which maps the RNS variables into super-Poincaré covariant Green-Schwarz-Siegel variables [9]. The untwisted $\hat{c} = 2$ $N = 2$ generators are defined in terms of the RNS fields as

$$T = T_{\rm RNS} - \frac{1}{2}\partial J, \quad G = j_{\rm BRST}, \quad \overline{G} = b, \quad J = cb + \eta\xi, \tag{7.1}$$

where $j_{\rm BRST}$ is the RNS BRST current and (ξ, η) come from fermionizing the (β, γ) ghosts as $\beta = \partial\xi e^{-\phi}$ and $\gamma = \eta e^{\phi}$. The field redefinition to Green-Schwarz-Siegel variables is then defined by

$$\theta^\alpha = e^{\frac{\phi}{2}}\Sigma^\alpha, \quad p_\alpha = e^{-\frac{\phi}{2}}\Sigma_\alpha \tag{7.2}$$

where Σ^α is the RNS spin field of conformal weight $\frac{5}{8}$. This field redefinition can be used for the subset of θ^α variables which are chosen in the $+\frac{1}{2}$ picture.

However, one can also consider the field redefinition [6]

$$\theta^a \equiv \widetilde{\psi}^a = \gamma\psi^a, \quad p_a \equiv \widetilde{\overline{\psi}}_a = (\gamma)^{-1}\overline{\psi}_a \tag{7.3}$$

where $a = 1$ to 5, $\psi^a = \frac{1}{\sqrt{2}}(\psi^a + i\psi^{a+5})$, and $\overline{\psi}_a = \frac{1}{\sqrt{2}}(\psi^a - i\psi^{a+5})$. This field redefinition is related to (5.1) by a fixed choice of λ^α in which the only nonzero component is λ^+. In terms of θ^a and p_a, the $\hat{c} = 2$ $N = 2$ generators of (7.1) are

$$T = -\partial x^a \partial \overline{x}_a - p_a \partial\theta^a + \widetilde{\beta}\partial\widetilde{\gamma} + \partial(\widetilde{\beta}\widetilde{\gamma}) - b\partial c - \partial(bc) - \frac{1}{2}\partial J, \tag{7.4}$$

$$G = c\left(T + \frac{1}{2}\partial J\right) + bc\partial c + \widetilde{\gamma} p_a \partial x^a + \theta^a \partial\overline{x}_a + \widetilde{\gamma} b, \tag{7.5}$$

$$\overline{G} = b, \tag{7.6}$$

$$J = cb + 2\widetilde{\beta}\widetilde{\gamma} + \theta^a p_a, \tag{7.7}$$

where $x^a = \frac{1}{\sqrt{2}}(x^a + ix^{a+5})$, $\overline{x}_a = \frac{1}{\sqrt{2}}(x^a - ix^{a+5})$, and $\widetilde{\gamma} = (\gamma)^2$. As in the hybrid formalism, all variables in (7.4) are automatically GSO-projected so there is no need to sum over spin structures.

Finally, performing the similarity transformation $\phi \to e^{-R}\phi e^R$ on all worldsheet fields ϕ where

$$R = \oint dz\ (cp_a\partial x^a + c\partial c\widetilde{\beta}), \tag{7.8}$$

one can express the $N = 2$ generators as [34]

$$T = -\partial x^a \partial \overline{x}_a - p_a \partial\theta^a + \widetilde{\beta}\partial\widetilde{\gamma} + \partial(\widetilde{\beta}\widetilde{\gamma}) - b\partial c - \partial(bc) - \frac{1}{2}\partial J, \tag{7.9}$$

$$G = \theta^a \partial\overline{x}_a + \widetilde{\gamma} b, \tag{7.10}$$

$$\overline{G} = -p_a \partial x^a + \widetilde{\beta}\partial c + \partial(\widetilde{\beta} c) + b, \tag{7.11}$$

$$J = \theta^a p_a + cb + 2\widetilde{\beta}\widetilde{\gamma}. \tag{7.12}$$

Note that the b ghost in $\overline{G}$ is not necessary for closure of the $N = 2$ superconformal algebra, and if one ignores the presence of the b ghost in $\overline{G}$, (7.9) are the standard $N = 2$ generators for a set of 5 chiral and antichiral scalar superfields, (x^a, θ^a) and $(\overline{x}_a, p_a)$, and a set of spin -1 chiral and spin 2 antichiral superfields, $(c, \widetilde{\gamma})$ and $(\widetilde{\beta}, b)$.

Since this version of the $N = 1 \to N = 2$ embedding contains similar fields to the $N = 1 \to N = 0$ map to the pure spinor formalism, it may be useful for proving the equivalence of the hybrid and pure spinor formalisms. Note that unlike the usual hybrid formalisms defined using (7.2), the $N = 2$ generators of (7.9) do not involve chiral bosons. For example, the $N = 2$ generators in the $d = 4$ hybrid formalism [29] involve a chiral boson ρ. These chiral bosons have been an obstacle to computing multiloop amplitudes using the hybrid formalism and it is possible this new version of the $N = 1 \to N = 2$ embedding will be useful for computing multiloop amplitudes which can be compared with the multiloop prescription in the pure spinor formalism.

Furthermore, one can easily compactify this version of the embedding on a Calabi-Yau D-fold by replacing $\sum_{a=1}^{5} \theta^a \partial \overline{x}_a$ with $\sum_{a=1}^{5-D} \theta^\alpha \partial \overline{x}_a + G_C$ and $\sum_{a=1}^{5} p_a \partial x^a$ with $\sum_{a=1}^{5-D} p_a \partial x^a + \overline{G}_C$ where $(G_C, \overline{G}_C)$ are the fermionic $N = 2$ generators of the Calabi-Yau D-fold. Finally, the fact that (x^a, θ^a) and $(\overline{x}_a, p_a)$ appear in the $N = 2$ generators of (7.9) in the same manner as they appear in topological strings suggests there may be a close connection between this version of the $N = 1 \to N = 2$ embedding and topological strings.

Acknowledgments

I would like to thank Yuri Aisaka, Osvaldo Chandía, Nikita Nekrasov and Warren Siegel for useful discussions, and CNPq grant 300256/94-9 and FAPESP grant 04/11426-0 for partial financial support.

References

[1] N. Berkovits, *Super-Poincaré covariant quantization of the superstring*, *JHEP* **04** (2000) 018 [hep-th/0001035].

[2] D. Friedan, E.J. Martinec and S.H. Shenker, *Conformal invariance, supersymmetry and string theory*, *Nucl. Phys.* **B 271** (1986) 93.

[3] N. Berkovits, *Pure spinor formalism as an $N = 2$ topological string*, *JHEP* **10** (2005) 089 [hep-th/0509120].

[4] N. Nekrasov, *Lectures at the* 23$^{\text{rd}}$ *Jerusalem winter school in theoretical physics*, January (2006), online at http://www.as.huji.ac.il/schools/phys23/media2.shtml.

[5] N. Berkovits and N. Nekrasov, *Multiloop superstring amplitudes from non-minimal pure spinor formalism*, *JHEP* **12** (2006) 029 [hep-th/0609012].

[6] L. Baulieu, *Transmutation of pure $2 - D$ supergravity into topological $2 - D$ gravity and other conformal theories*, *Phys. Lett.* **B 288** (1992) 59 [hep-th/9206019];
L. Baulieu, M.B. Green and E. Rabinovici, *A unifying topological action for heterotic and type-II superstring theories*, *Phys. Lett.* **B 386** (1996) 91 [hep-th/9606080];
L. Baulieu and N. Ohta, *Worldsheets with extended supersymmetry*, *Phys. Lett.* **B 391** (1997) 295 [hep-th/9609207];
L. Baulieu, M.B. Green and E. Rabinovici, *Superstrings from theories with $N > 1$ world-sheet supersymmetry*, *Nucl. Phys.* **B 498** (1997) 119 [hep-th/9611136].

[7] N. Berkovits and C. Vafa, *On the uniqueness of string theory*, *Mod. Phys. Lett.* **A 9** (1994) 653 [hep-th/9310170].

[8] J. Hoogeveen and K. Skenderis, *BRST quantization of the pure spinor superstring*, *JHEP* **11** (2007) 081 [arXiv:0710.2598].

[9] N. Berkovits, *A new description of the superstring*, hep-th/9604123.

[10] W. Siegel, *Classical superstring mechanics*, *Nucl. Phys.* **B 263** (1986) 93.

[11] N. Berkovits, *Relating the RNS and pure spinor formalisms for the superstring*, *JHEP* **08** (2001) 026 [hep-th/0104247].

[12] N.A. Nekrasov, *Lectures on curved beta-gamma systems, pure spinors and anomalies*, hep-th/0511008.

[13] P. Mukhopadhyay, *DDF construction and D-brane boundary states in pure spinor formalism*, *JHEP* **05** (2006) 055 [hep-th/0512161].

[14] R. Schiappa and N. Wyllard, *D-brane boundary states in the pure spinor superstring*, *JHEP* **07** (2005) 070 [hep-th/0503123].

[15] K. Lee and W. Siegel, *Simpler superstring scattering*, *JHEP* **06** (2006) 046 [hep-th/0603218].

[16] N. Berkovits, *Multiloop amplitudes and vanishing theorems using the pure spinor formalism for the superstring*, *JHEP* **09** (2004) 047 [hep-th/0406055].

[17] L. Brink and J.H. Schwarz, *Quantum superspace*, *Phys. Lett.* **B 100** (1981) 310.

[18] W. Siegel, *Hidden local supersymmetry in the supersymmetric particle action*, *Phys. Lett.* **B 128** (1983) 397.

[19] M.B. Green and J.H. Schwarz, *Covariant description of superstrings*, *Phys. Lett.* **B 136** (1984) 367.

[20] N. Berkovits and B.C. Vallilo, *Consistency of super-Poincaré covariant superstring tree amplitudes*, *JHEP* **07** (2000) 015 [hep-th/0004171].

[21] H. Ishikawa and M. Kato, *Note on $N = 0$ string as $N = 1$ string*, *Mod. Phys. Lett.* **A 9** (1994) 725 [hep-th/9311139].

[22] P.A. Grassi, G. Policastro, M. Porrati and P. Van Nieuwenhuizen, *Covariant quantization of superstrings without pure spinor constraints*, *JHEP* **10** (2002) 054 [hep-th/0112162]; P.A. Grassi, G. Policastro and P. van Nieuwenhuizen, *The quantum superstring as a WZNW model*, *Nucl. Phys.* **B 676** (2004) 43 [hep-th/0307056].

[23] Y. Aisaka and Y. Kazama, *A new first class algebra, homological perturbation and extension of pure spinor formalism for superstring*, *JHEP* **02** (2003) 017 [hep-th/0212316]; *Relating Green-Schwarz and extended pure spinor formalisms by similarity transformation*, *JHEP* **04** (2004) 070 [hep-th/0404141].

[24] M. Chesterman, *Ghost constraints and the covariant quantization of the superparticle in ten dimensions*, *JHEP* **02** (2004) 011 [hep-th/0212261].

[25] K. Lee and W. Siegel, *Conquest of the ghost pyramid of the superstring*, *JHEP* **08** (2005) 102 [hep-th/0506198].

[26] N. Berkovits and D.Z. Marchioro, *Relating the Green-Schwarz and pure spinor formalisms for the superstring*, *JHEP* **01** (2005) 018 [hep-th/0412198].

[27] Y. Aisaka and Y. Kazama, *Origin of pure spinor superstring*, *JHEP* **05** (2005) 046 [hep-th/0502208].

[28] M. Matone, L. Mazzucato, I. Oda, D. Sorokin and M. Tonin, *The superembedding origin of the Berkovits pure spinor covariant quantization of superstrings*, *Nucl. Phys.* **B 639** (2002) 182 [hep-th/0206104].

[29] N. Berkovits, *Covariant quantization of the Green-Schwarz superstring in a Calabi-Yau background*, *Nucl. Phys.* **B 431** (1994) 258 [hep-th/9404162].

[30] M. Tonin, *World sheet supersymmetric formulations of Green-Schwarz superstrings*, *Phys. Lett.* **B 266** (1991) 312.

[31] N. Berkovits, *The heterotic Green-Schwarz superstring on an $N = (2,0)$ superworldsheet*, *Nucl. Phys.* **B 379** (1992) 96 [hep-th/9201004].

[32] N. Berkovits, *SuperPoincaré invariant superstring field theory*, *Nucl. Phys.* **B 450** (1995) 90 [hep-th/9503099].

[33] O. Chandía, private communication.

[34] N. Berkovits, *Strings* 2002 *proceedings*, University of Cambridge, online at http://www.damtp.cam.ac.uk/strings02/avt/berkovits/img_16.html.